C4 JUNE 11

1. $$\frac{9x^2}{(x-1)^2(2x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(2x+1)}$$

Find the values of the constants A, B and C.

$\Rightarrow \; 9x^2 \equiv A(x-1)(2x+1) + B(2x+1) + C(x-1)^2$

$x=1 \;\Rightarrow\; 9 \equiv 3B \;\Rightarrow\; \underline{B=3}$

$x=-\frac{1}{2} \;\Rightarrow\; \frac{9}{4} \equiv \frac{9}{4}C \;\Rightarrow\; \underline{C=1}$

$x=0 \quad 0 \equiv -A+B+C \;\Rightarrow\; -A+3+1=0 \;\Rightarrow\; \underline{A=4}$

2. $$f(x) = \frac{1}{\sqrt{(9+4x^2)}}, \qquad |x| < \frac{3}{2}$$

Find the first three non-zero terms of the binomial expansion of $f(x)$ in ascending powers of x. Give each coefficient as a simplified fraction.

$$f(x) = (9+4x^2)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}\left(1 + \frac{4}{9}x^2\right)^{-\frac{1}{2}}$$

$$f(x) \simeq \frac{1}{3}\left(1 + \left(-\frac{1}{2}\right)\left(\frac{4}{9}x^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{4}{9}x^2\right)^2\right)$$

$$\simeq \frac{1}{3}\left(1 - \frac{2}{9}x^2 + \frac{2}{27}x^4\right) \simeq \frac{1}{3} - \frac{2}{27}x^2 + \frac{2}{81}x^4$$

3.

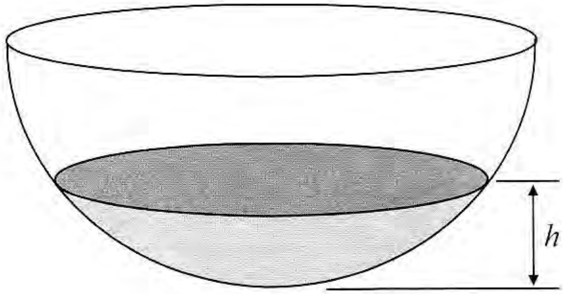


Figure 1

A hollow hemispherical bowl is shown in Figure 1. Water is flowing into the bowl. When the depth of the water is h m, the volume $V\,\text{m}^3$ is given by

$$V = \frac{1}{12}\pi h^2(3-4h), \qquad 0 \leqslant h \leqslant 0.25$$

(a) Find, in terms of π, $\dfrac{dV}{dh}$ when $h = 0.1$

(4)

Water flows into the bowl at a rate of $\dfrac{\pi}{800}\ \text{m}^3\text{s}^{-1}$.

(b) Find the rate of change of h, in m s^{-1}, when $h = 0.1$

(2)

a) $V = \frac{1}{12}\pi h^2(3-4h) = \frac{1}{4}\pi h^2 - \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = \frac{1}{2}\pi h - \pi h^2 \qquad h = 0.1 \Rightarrow \frac{dV}{dh} = \frac{1}{20}\pi - \frac{1}{100}\pi$

$\Rightarrow \frac{dV}{dh} = \frac{\pi}{25}$

b) $\frac{dV}{dt} = \frac{\pi}{800} \qquad \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{25}{\cancel{\pi}} \times \frac{\cancel{\pi}}{800} = \frac{1}{32}$ when $h = 0.1$

4.

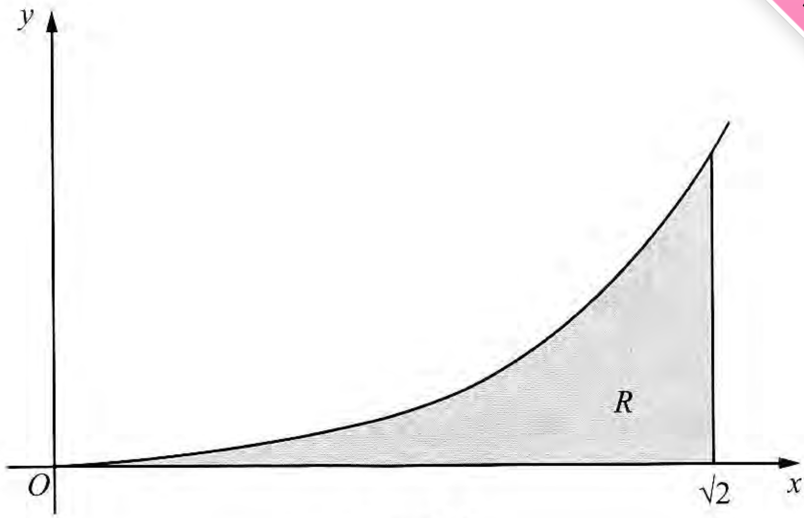


Figure 2

Figure 2 shows a sketch of the curve with equation $y = x^3 \ln(x^2 + 2)$, $x \geqslant 0$.
The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the line $x = \sqrt{2}$.

The table below shows corresponding values of x and y for $y = x^3 \ln(x^2 + 2)$.

x	0	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{4}$	$\sqrt{2}$
y	0	0.0333	0.3240	1.3596	3.9210

(a) Complete the table above giving the missing values of y to 4 decimal places.

(2)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places.

(3)

(c) Use the substitution $u = x^2 + 2$ to show that the area of R is

$$\frac{1}{2}\int_2^4 (u-2)\ln u \, du$$

(4)

(d) Hence, or otherwise, find the exact area of R.

(6)

b) $\frac{1}{2}\left(\frac{\sqrt{2}}{4}\right)\left(0+3.9210+2\left(0.0333+0.324+\right.\right.$

$\approx \underline{1.30}$ (2dp)

c) $u = x^2+2$ $\qquad x=0 \quad u=0^2+2=2$

$\frac{du}{dx} = 2x$ $\qquad x=\sqrt{2} \quad u=(\sqrt{2})^2+2=4$

$dx = \frac{du}{2x}$ $\qquad x^2 = u-2$

$$\int_0^{\sqrt{2}} x^3\ln(x^2+2)\,dx = \int_2^4 x^3 \ln u \,\frac{du}{2x}$$

$$= \frac{1}{2}\int_2^4 x^2 \ln u \,du = \frac{1}{2}\int_2^4 (u-2)\ln u \,du$$

d) $R = \frac{1}{2}\int_2^4 (u-2)\ln u \,du$ $\qquad u = \ln u \quad \frac{dv}{du} = u-2$

$\frac{du}{du}^{*} = \frac{1}{u} \quad v = \frac{1}{2}u^2-2u$

$$R = \frac{1}{2}\left((\tfrac{1}{2}u^2-2u)\ln u - \int (\tfrac{1}{2}u^2-2u)\tfrac{1}{u}\,du \right)$$

$$R = \frac{1}{2}\left((\tfrac{1}{2}u^2-2u)\ln u - \int \tfrac{1}{2}u - 2\,du \right)$$

$$R = \frac{1}{2}\left[(\tfrac{1}{2}u^2-2u)\ln u - \tfrac{1}{4}u^2 + 2u \right]_2^4$$

$$R = \frac{1}{2}\left[(0-4+8) - (-2\ln 2 - 1 + 4) \right]$$

$$R = \frac{1}{2}\left[1 + 2\ln 2 \right] = \frac{1}{2} + \ln 2$$

5. Find the gradient of the curve with equation

$$\ln y = 2x \ln x, \qquad x > 0,\ y > 0$$

at the point on the curve where $x = 2$. Give your answer as an exact value.

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(2x\ln x)$ $\qquad u = 2x \quad v = \ln x$

$\qquad u' = 2 \quad v' = \frac{1}{x}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2\ln x + 2$ $\qquad vu' + uv'$

$\Rightarrow \frac{dy}{dx} = y(2\ln x + 2)$

when $x=2$ $\quad \ln y = 4\ln 2 \Rightarrow \ln y = \ln 16 \Rightarrow y = 16$

$\therefore \left.\frac{dy}{dx}\right|_{x=2} = 16(2\ln 2 + 2) = \underline{32\ln 2 + 32}$

6. With respect to a fixed origin O, the lines l_1 and l_2 are given by the eq

$$l_1: \quad \mathbf{r} = \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \qquad l_2: \quad \mathbf{r} = \begin{pmatrix} -5 \\ 15 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

where λ and μ are scalar parameters.

(a) Show that l_1 and l_2 meet and find the position vector of their point of intersection A.
(6)

(b) Find, to the nearest 0.1°, the acute angle between l_1 and l_2.
(3)

The point B has position vector $\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$.

(c) Show that B lies on l_1.
(1)

(d) Find the shortest distance from B to the line l_2, giving your answer to 3 significant figures.
(4)

a) $\begin{pmatrix} 6-\lambda \\ -3+2\lambda \\ -2+3\lambda \end{pmatrix} = \begin{pmatrix} -5+2\mu \\ 15-3\mu \\ 3+\mu \end{pmatrix} \Rightarrow \begin{array}{l} 6-\lambda = -5+2\mu \quad - i \\ -3+2\lambda = 15-3\mu - j \\ -2+3\lambda = 3+\mu - k \end{array}$

i) $\Rightarrow \lambda = 11-2\mu$ into j) $-3+22-4\mu = 15-3\mu$

$\Rightarrow \mu = 4, \lambda = 3$

check with k $\Rightarrow -2+3\lambda = 7 \quad 3+\mu = 7 \quad \therefore$ they meet

$\lambda = 3 \Rightarrow A \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}$

b) $\theta = \cos^{-1}\left(\left| \dfrac{\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right|} \right| \right) \quad \theta = \cos^{-1}\left(\dfrac{|-5|}{\sqrt{14}\sqrt{14}}\right)$

$\theta = \cos^{-1}\left(\dfrac{5}{14}\right)$

$\underline{\theta = 69.1°}$

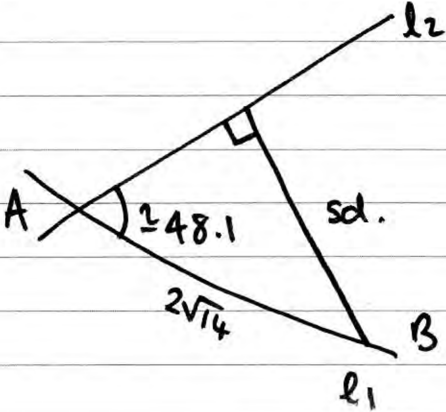
c) $\begin{pmatrix} 6-\lambda \\ -3+2\lambda \\ -2+3\lambda \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

$6-\lambda = 5 \Rightarrow \lambda = 1$

$-3+2\lambda = -1 \Rightarrow 2\lambda = 2$

$-2+3\lambda = 1 \Rightarrow 3\lambda = 3 \Rightarrow$

d)



$\overrightarrow{AB} = b - a = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{2^2+4^2+6^2} = 2\sqrt{14}$

$\therefore sd = 2\sqrt{14}\sin(69.1\ldots) = \underline{6.99}$ (3sf)

7.

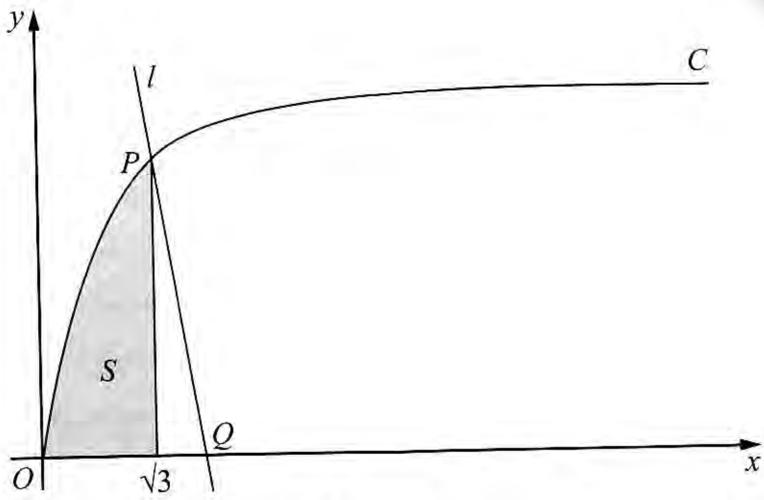


Figure 3

Figure 3 shows part of the curve C with parametric equations

$$x = \tan\theta\,, \qquad y = \sin\theta\,, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point P lies on C and has coordinates $\left(\sqrt{3}, \frac{1}{2}\sqrt{3}\right)$.

(a) Find the value of θ at the point P.

(2)

The line l is a normal to C at P. The normal cuts the x-axis at the point Q.

(b) Show that Q has coordinates $(k\sqrt{3}, 0)$, giving the value of the constant k.

(6)

The finite shaded region S shown in Figure 3 is bounded by the curve C, the line $x = \sqrt{3}$ and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the volume of the solid of revolution, giving your answer in the form $p\pi\sqrt{3} + q\pi^2$, where p and q are constants.

(7)

a) $x = \sqrt{3} \Rightarrow \sqrt{3} = \tan\theta \Rightarrow \underline{\theta = \frac{\pi}{3}}$

b) $\frac{dx}{d\theta} = \sec^2\theta \quad \frac{dy}{d\theta} = \cos\theta \Rightarrow \frac{dy}{dx} = \frac{\cos\theta}{\sec^2\theta}$

$\theta = \frac{\pi}{3}$ at P $\quad \frac{dy}{dx} = \left(\cos\left(\frac{\pi}{3}\right)\right)^3 = \frac{1}{8} \Rightarrow m_n = -8$

$\Rightarrow y - \frac{\sqrt{3}}{2} = -8(x - \sqrt{3})$ Crosses x when $y = 0$

$\Rightarrow -\frac{\sqrt{3}}{2} = -8(x-\sqrt{3}) \Rightarrow \frac{\sqrt{3}}{16} = x - \sqrt{3} \Rightarrow \underline{x = \frac{17}{16}\sqrt{3}}$

c) $\text{Vol} = \pi\int_0^{\sqrt{3}} y^2 \frac{dx}{d\theta}\, d\theta = \pi\int_0^{\frac{\pi}{3}} \sin^2\theta \times \sec^2\theta \, d\theta$

$= \pi\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = \pi\int_0^{\frac{\pi}{3}} \tan^2\theta \, d\theta$

$\frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2}$

$\tan^2\theta + 1 = \sec^2\theta$

$= \pi\int_0^{\frac{\pi}{3}} \sec^2\theta - 1 \, d\theta$

$= \pi\left[\tan\theta - \theta\right]_0^{\frac{\pi}{3}} = \pi\left[\left(\sqrt{3} - \frac{\pi}{3}\right) - (0-0)\right]$

$\underline{= \pi\sqrt{3} - \frac{1}{3}\pi^2}$

8. (a) Find $\int (4y+3)^{-\frac{1}{2}}\,dy$

(b) Given that $y = 1.5$ at $x = -2$, solve the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$$

giving your answer in the form $y = f(x)$.

(6)

a) $x = (4y+3)^{\frac{1}{2}} \quad \times\left(\times\frac{1}{2}\right)$

$\frac{dx}{dy} = \frac{1}{2}(4y+3)^{-\frac{1}{2}} \times 4 \quad \left(\times\frac{1}{2}\right)$

$\therefore \int (4y+3)^{-\frac{1}{2}}\,dy = \frac{1}{2}(4y+3)^{\frac{1}{2}} + c$

b) $\int (4y+3)^{-\frac{1}{2}}\,dy = \int \frac{1}{x^2}\,dx$

$\Rightarrow \frac{1}{2}(4y+3)^{\frac{1}{2}} + c = -\frac{1}{x}$

$(-2, 1\cdot5) \Rightarrow \frac{3}{2} + c = \frac{1}{2} \Rightarrow c = -1$

$\Rightarrow \frac{1}{2}(4y+3)^{\frac{1}{2}} - 1 = -\frac{1}{x} \Rightarrow (4y+3)^{\frac{1}{2}} = 2 - \frac{2}{x}$

$\Rightarrow 4y+3 = \left(2 - \frac{2}{x}\right)^2 \Rightarrow y = \frac{1}{4}\left(2 - \frac{2}{x}\right)^2 - \frac{3}{4}$